Journal of Algebra 321 (2009) 548–557

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Injectivity relative to closed submodules

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ARTICLE INFO

Article history:
Received 5 February 2008
Available online 25 November 2008
Communicated by J.T. Stafford

Keywords:
Injective module
Closed submodule
Dedekind domain
Primitive ideal

ABSTRACT

Let R be a ring. An R-module X is called c-injective if, for every closed submodule L of every R-module M, every homomorphism from L to X lifts to M. It is proved that if R is a Dedekind domain then an R-module X is c-injective if and only if X is isomorphic to a direct product of homogeneous semisimple R-modules and injective R-modules. It is also proved that a commutative Noetherian domain R is Dedekind if and only if every simple R-module is c-injective.

1. Introduction

We shall assume that all rings are associative with identity and all modules are unitary left modules. Let R be any ring. A submodule K of an R-module M is called *closed* (*in* M) provided K has no proper essential extension in M. Clearly, every direct summand of M is closed in M. Moreover, if L is any submodule of M then there exists, by Zorn's Lemma, a submodule K of M maximal with respect to the property that L is an essential submodule of K and in this case K is a closed submodule of M. A module M is called an *extending* module if every closed submodule is a direct summand, and in this case every submodule of M is essential in a direct summand of M. For the properties of closed submodules and extending modules see [3] or [14].

Let M be any R-module. In [16] an R-module X is called M-*c*-*injective* provided, for every closed submodule K of M, every homomorphism $\varphi : K \to X$ can be lifted to a homomorphism $\theta : M \to X$. Moreover, X is called *c*-*injective* provided X is M-c-injective for every R-module M. Note that if M is an extending module then every R-module is M-c-injective. It is proved in [17, Theorem 6] that if R is a Dedekind domain and an R-module M is a direct product of simple R-modules then M is

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.11.004

M-c-injective but M need not be an extending module (see [17, Proposition 2]). Firstly, we prove that if R is a Dedekind domain then an R-module X is c-injective if and only if there exists an R-module Y such that Y is a direct product of simple R-modules and an injective R-module with the property that X is isomorphic to a direct summand of Y. We then show that such a direct summand is isomorphic to a direct product of homogeneous semisimple R-modules and injective R-modules. For related material see also [2,18].

Let R be any ring and let $\mathcal{E}$ be any non-empty collection of ideals of R. Following [11], a submodule L of an R-module M is called *$\mathcal{E}$-pure* in M provided $L \cap IM = IL$ for every ideal I in $\mathcal{E}$ (see also [13]). We shall call an R-module X *$\mathcal{E}$-pure-injective* provided, for every R-module M and every $\mathcal{E}$-pure submodule L of M, every homomorphism $\varphi : L \to X$ lifts to M. In particular, we shall be interested in these concepts in the case $\mathcal{E}$ is the collection of left primitive ideals of R. We shall denote the collection of left primitive ideals of R by $\mathcal{P}$.

Recall that Honda [9, pp. 42–43] defines a submodule B of an Abelian group A to be *neat* provided $pB = B \cap pA$ for all primes p (see also [5]). Thus a submodule B of a $\mathbb{Z}$-module A is neat if and only if B is a $\mathcal{P}$-pure submodule of A (in our terminology).

We shall characterize c-injective modules over a Dedekind domain R by first characterizing $\mathcal{P}$-pure-injective modules over a large class of rings and by then showing that for R the class of c-injective modules is precisely the class of $\mathcal{P}$-pure-injective modules. Finally we shall show that a commutative Noetherian domain R is Dedekind if and only if every simple R-module is c-injective.

2. $\mathcal{P}$-pure-injective modules

In this section, our aim is to characterize $\mathcal{P}$-pure-injective R-modules for a large class of rings R. We begin with the following elementary result which is included for completeness.

Lemma 2.1. *Let K be a submodule of a module M, let X be a module and let $\varphi : K \to X$ be a homomorphism such that $K/\ker\varphi$ is a direct summand of the module $M/\ker\varphi$. Then φ can be lifted to a homomorphism $\theta : M \to X$.*

Proof. Let $L = \ker\varphi$. There exists a submodule H of M containing L such that $M = K + H$ and $K \cap H = L$. Define a mapping $\theta : M \to X$ by $\theta(x + y) = \varphi(x)$ for all $x \in K$, $y \in H$. It is easy to check that θ is well defined and is a homomorphism which lifts φ to M. □

Corollary 2.2. *Let R be any ring, let K be a submodule of an R-module M, let X be an R-module and let A be an ideal of R such that $AX = 0$. If K/AK is a direct summand of the module M/AK then every homomorphism $\varphi : K \to X$ can be lifted to M.*

Proof. Let $L = \ker\varphi$. Then K/L embeds in X and hence $A(K/L) = 0$, i.e. $AK \subseteq L$. Since K/AK is a direct summand of M/AK it follows that K/L is a direct summand of M/L. The result follows by Lemma 2.1. □

Next we give a further consequence of Lemma 2.1.

Lemma 2.3. *Let K be a submodule of a module M and let X be a semisimple module. Then the following statements are equivalent.*

(i) *Every homomorphism $\varphi : K \to X$ can be lifted to a homomorphism $\theta : M \to X$.*
(ii) *K/L is a direct summand of M/L for every submodule L of K such that K/L is isomorphic to a submodule of X.*

Proof. (i) $\Rightarrow$ (ii). Let L be any submodule of K such that K/L is isomorphic to a submodule Y of X and let $\alpha : K/L \to Y$ be an isomorphism. Note that Y is a direct summand of X. Let $\pi : X \to Y$ denote the canonical projection. Consider the mapping $\lambda : K \to X$ defined by $\lambda(m) = \alpha(m + L)$ for all $m \in K$.

By hypothesis, there exists a homomorphism $\mu : M \to X$ such that $\mu|_K = \lambda$. Let $\nu = \pi\mu$. If $z \in M$ then $\nu(z) \in Y$ so that $\nu(z) = \alpha(y + L) = \lambda(y) = \nu(y)$, for some y in K. It follows that $M = K + \ker\nu$. In addition, $K \cap \ker\nu = \ker\lambda = L$. Thus $M/L = (K/L) \oplus (\ker\nu)/L$.

(ii) $\Rightarrow$ (i). By Lemma 2.1. □

Lemma 2.4. *Let R be a ring and let S be an ideal of R such that the ring R/S is semiprime Artinian. Then the following statements are equivalent for a submodule K of an R-module M.*

(i) $SK = K \cap SM$.
(ii) *K/SK is a direct summand of the module M/SK.*
(iii) *For every (R/S)-module X, every homomorphism $\varphi : K \to X$ can be lifted to M.*
(iv) *Every R-homomorphism $\varphi : K \to R/S$ can be lifted to M.*
(v) *For every simple (R/S)-module U, every R-homomorphism $\varphi : K \to U$ can be lifted to M.*

Proof. (i) $\Rightarrow$ (ii). Because R/S is a semiprime Artinian ring, the module M/SM is semisimple and hence there exists a submodule H of M containing SM such that

$$M/SM = \big[(K + SM)/SM\big] \oplus (H/SM).$$

Then (i) gives that $M/SK = (K/SK) \oplus (H/SK)$.

(ii) $\Rightarrow$ (iii). By Corollary 2.2.

(iii) $\Rightarrow$ (iv). Clear.

(iv) $\Rightarrow$ (v). Clear because R/S being semiprime Artinian gives that every simple (R/S)-module is a direct summand of R/S.

(v) $\Rightarrow$ (i). Suppose that $SK \neq K \cap SM$. Because R/S is semiprime Artinian, the module K/SK is semisimple and hence there exists a proper submodule T of K containing SK such that

$$K/SK = \big[(K \cap SM)/SK\big] \oplus (T/SK).$$

Because K/T is a non-zero (R/S)-module, there exists a maximal submodule N of K containing T. Note that K/N is a simple (R/S)-module. Let $\alpha : K \to K/N$ denote the canonical projection with kernel N. By hypothesis α can be lifted to a non-zero homomorphism $\beta : M \to K/N$. Let $H = \ker\beta$. Note that $M/H \cong K/N$ so that H is a maximal submodule of M. Note further that $H \cap K = \ker\alpha = N$. Thus $K \nsubseteq H$ and hence $M = H + K$. Now $SM \subseteq H$ and hence $K \cap SM \subseteq K \cap H = N$. But $K = (K \cap SM) + T \subseteq N$, so that $K = N$, a contradiction. Thus $SK = K \cap SM$. □

A module M will be called a *homogeneous semisimple* module provided there exists a simple module U such that M is isomorphic to a direct sum of copies of U.

Corollary 2.5. *Let R be a ring and let U be a homogeneous semisimple R-module with annihilator P in R such that the ring R/P is* (*simple*) *Artinian. Then the following statements are equivalent for a submodule K of an R-module M.*

(i) $PK = K \cap PM$.
(ii) *Every homomorphism $\varphi : K \to U$ can be lifted to M.*

Proof. By Lemma 2.4. □

In view of Corollary 2.5 we are interested in rings R such that R/P is an Artinian ring for every left primitive ideal P of R. Commutative rings clearly have this property. More generally rings satisfying a polynomial identity also satisfy this property by a theorem of Kaplansky (see, for example, [12, Theorem 13.3.8]). Recall that a ring R is called *left fully bounded* provided, for each prime homomorphic image of R, every essential left ideal contains a non-zero two-sided ideal. Recall further that a ring R is called a *left FBN ring* if R is a left fully bounded left Noetherian ring. It is well known that if R is a left FBN ring then R/P is an Artinian ring for every left primitive ideal P of R (see, for example, [7, Proposition 8.4]). Note too that if R is a semiperfect ring then R/P is Artinian for every left primitive ideal P of R. Finally, Roseblade [15, Corollary A] proves that if $J = \mathbb{Z}$ or J is a finite field, G is a polycyclic-by-finite group and R is the group ring $J[G]$ then R/P is Artinian for every left primitive ideal P of R. For all these rings we have the following result.

Corollary 2.6. *Let R be a ring such that R/P is an Artinian ring for every left primitive ideal P of R. Then the following statements are equivalent for a submodule N of an R-module M.*

(i) *N is a $\mathcal{P}$-pure submodule of M.*
(ii) *For every simple R-module U, every homomorphism $\varphi : N \to U$ can be lifted to M.*
(iii) *For every homogeneous semisimple R-module X, every homomorphism $\varphi : N \to X$ can be lifted to M.*

Proof. By Corollary 2.5. $\square$

In particular, Corollary 2.6 shows that if R is a ring such that R/P is Artinian for every left primitive ideal P then every simple R-module is $\mathcal{P}$-pure-injective. Moreover, Corollary 2.6 allows us to characterize $\mathcal{P}$-pure-injective modules over such rings R. First we prove a lemma.

Lemma 2.7. *Let R be a ring such that R/P is an Artinian ring for every left primitive ideal P.*

(i) *Every direct product of $\mathcal{P}$-pure injective R-modules is $\mathcal{P}$-pure-injective.*
(ii) *Every direct summand of a $\mathcal{P}$-pure-injective R-module is $\mathcal{P}$-pure-injective.*
(iii) *For every R-module X there exists a $\mathcal{P}$-pure-injective R-module Y such that X is isomorphic to a $\mathcal{P}$-pure submodule of Y.*

Proof. (i), (ii) Standard.

(iii) Let $\mathcal{M}$ denote the collection of all maximal left ideals of R. For each V in $\mathcal{M}$, let $\mathcal{F}_V$ denote the set of all non-zero R-module homomorphisms from X to R/V. Let $\mathcal{F} = \bigcup_{V \in \mathcal{M}} \mathcal{F}_V$. Index the set $\mathcal{F}$ by a set I, so that $\mathcal{F}$ is the collection of all non-zero homomorphisms $\varphi_i : X \to U_i$ where, for each $i \in I$, $U_i \cong R/V$ for some V in $\mathcal{M}$. Let E denote the injective envelope of X and let Y denote the direct product $E \times \prod_{i \in I} U_i$. By Corollary 2.6, U_i is $\mathcal{P}$-pure-injective for every $i \in I$ and hence, by (i), Y is $\mathcal{P}$-pure-injective. Define a mapping $\theta : X \to Y$ by $\theta(x) = (x, (\varphi_i(x))_{i \in I})$ for all x in X. Clearly θ is an R-monomorphism. Let $X' = \theta(X)$. If U is any simple R-module and $\alpha : X' \to U$ any homomorphism then $\alpha\theta : X \to U$ is a homomorphism so that $\alpha\theta = \varphi_j$ for some $j \in I$. In this case $U = U_j$. Define a mapping $\overline{\beta} : Y \to U_j$ by $\overline{\beta}(e, (u_i)_{i \in I}) = u_j$ for all $e \in E$ and $u_i \in U_i$ $(i \in I)$. It is easy to check that $\overline{\beta} : Y \to U$ is a homomorphism that extends α. Again using Corollary 2.6, we see that X' is a $\mathcal{P}$-pure submodule of Y. $\square$

Theorem 2.8. *Let R be a ring such that R/P is Artinian for every left primitive ideal P. Then an R-module X is $\mathcal{P}$-pure-injective if and only if X is isomorphic to a direct summand of a module Y where Y is a direct product of simple modules and injective modules.*

Proof. The 'if' part follows from Lemma 2.7(i) and (ii) since every simple R-module is $\mathcal{P}$-pure-injective by Corollary 2.6. Conversely suppose X is $\mathcal{P}$-pure-injective. By Lemma 2.7(iii), there exists a $\mathcal{P}$-pure-injective R-module Y such that X is isomorphic to a $\mathcal{P}$-pure submodule X' of Y. But then the $\mathcal{P}$-pure-injective X' which is a $\mathcal{P}$-pure submodule of Y must be a direct summand of Y. $\square$

3. Supplement submodules

There is an important class of submodules of an R-module M which are all $\mathcal{P}$-pure in M for rings R with the property that R/P is Artinian for every P in $\mathcal{P}$. Recall that if R is any ring, a submodule L of an R-module M is called a *supplement in* M provided there exists a submodule N of M such that $M = N + L$ and L is minimal with respect to this property. It is well known (and easy to prove) that L is a supplement in M if and only if there exists a submodule N of M such that $M = N + L$ and $N \cap L$ is small in L.

Lemma 3.1. *Let R be a ring and let S be an ideal of R such that the ring R/S is semiprime Artinian. Let L be a supplement of an R-module M. Then L/SL is a direct summand of the R-module M/SL.*

Proof. There exists a submodule N of M such that $M = N + L$ and $N \cap L$ is small in L. Note that the module L/SL is semisimple and hence has zero (Jacobson) radical. It follows that $N \cap L \subseteq SL$. Hence $M/SL = [(N + SL)/SL] \oplus (L/SL)$. $\square$

Corollary 3.2. *Let R be a ring such that R/P is Artinian for every left primitive ideal P of R and let M be any R-module. Then every supplement in M is a $\mathcal{P}$-pure submodule of M.*

Proof. By Lemmas 2.4 and 3.1. $\square$

In certain circumstances the converse of Corollary 3.2 is true. We mention one. First we prove a simple lemma.

Lemma 3.3. *Let S be an ideal of a ring R such that R/S is semiprime Artinian and let L be a $\mathcal{P}$-pure submodule of an R-module M. Then $SL = L \cap SM$.*

Proof. It is clearly sufficient to prove that if $S = A \cap B$ for some ideals A and B of R such that $R = A + B$, $AL = L \cap AM$ and $BL = L \cap BM$ then $SL = L \cap SM$. Now $L \cap SM$ is contained in $L \cap AM = AL$ and similarly $L \cap SM \subseteq BL$. Thus $L \cap SM \subseteq AL \cap BL$. But $AL \cap BL = (A + B)(AL \cap BL) \subseteq ABL + BAL \subseteq SL$. Thus $L \cap SM \subseteq SL$ and it follows at once that $L \cap SM = SL$. $\square$

Proposition 3.4. *Let R be a left perfect ring. Then a submodule L of an R-module M is a supplement in M if and only if L is a $\mathcal{P}$-pure submodule of M.*

Proof. The necessity follows by Corollary 3.2. Conversely, suppose that L is a $\mathcal{P}$-pure submodule of M. Let S denote the Jacobson radical of R. By Lemma 3.3 $L \cap SM = SL$ and hence L/SL is a direct summand of M/SL by Lemma 2.4. Let N be a submodule of M such that $M = N + L$ and $N \cap L = SL$. We always have $SL \leqslant \mathrm{Rad}(L)$ by for example [1, Proposition 15.18]. Since the ring R is left perfect, every R-module contains a maximal submodule and so $\mathrm{Rad}(L)$ is the largest small submodule of L by for example [1, Proposition 9.18]. Thus SL is a small submodule of L and hence L is a supplement in M. $\square$

4. Almost principal ideals

In this section we shall investigate the relationship between the closed submodules and the $\mathcal{P}$-pure submodules of an R-module.

Let R be a commutative ring with identity. An ideal P of R will be called *almost principal* provided there exist elements a, b in P such that $(1 - a)P \subseteq Rb$. Note that if R is a domain then a maximal ideal P of R is almost principal if and only if P is invertible (see, for example, [4, Theorem 1.2]). In particular, every maximal ideal of a Dedekind domain is almost principal (see, for example, [20, Theorem 10, p. 273]). More generally, let S be a Dedekind domain and let X be a non-zero injective R-module. Let R denote the trivial extension of X by S, that is R consists of all elements (s, x) with s in S and x in X with addition and multiplication given by

$$(s, x) + (s', x') = (s + s', x + x')$$

and

$$(s, x)(s', x') = (ss', sx' + s'x)$$

for all $s, s' \in S$, $x, x' \in X$. Then R is a commutative ring which is neither semiprime nor Noetherian but every maximal ideal of R is almost principal.

In this section we shall prove the following result.

Proposition 4.1. *Let R be a commutative ring and let K be a submodule of an R-module M. If K is a closed submodule of M then $PK = K \cap PM$ for every almost principal maximal ideal P of R. Moreover, the converse holds in case every essential ideal of R contains a (finite) product of almost principal maximal ideals.*

To prove the result we require the following lemmas.

Lemma 4.2. *Let P be a principal maximal ideal of a commutative ring R. Then the R-module R/P^n is uniserial for every positive integer n.*

Proof. Standard. □

Corollary 4.3. *Let P be an almost principal maximal ideal of a commutative ring R. Then the R-module R/P^n is uniserial for every positive integer n.*

Proof. Let n be any positive integer. There exist elements a, b in P such that $(1-a)P \subseteq Rb$. Let $x \in P$. Then $(1-a)x = rb$ for some r in R. It follows that $x \in Rb + P^2$. Hence $P = Rb + P^2$. By a standard argument $P = Rb + P^n$. By passing to the ring R/P^n, Lemma 4.2 gives the required result. □

Lemma 4.4. *Let R be a commutative ring which contains an almost principal maximal ideal P. Let K be a closed submodule of an R-module M. Then $PK = K \cap PM$.*

Proof. There exist elements a, b in P such that $(1-a)P \subseteq Rb$. Note that $PK \subseteq K \cap PM$ and that K/PK is a closed submodule of the module M/PK, so that without loss of generality we can suppose that $PK = 0$. Suppose that $K \cap PM$ is non-zero and let x be any non-zero element of $K \cap PM$. Then $x = (1-a)x \in (1-a)PM \subseteq bM$, so that there exists $m \in M$ such that $x = bm$. Now $Pbm = Px \subseteq PK = 0$. It follows that $P^2(1-a)m = 0$. By Corollary 4.3, the R-module $R(1-a)m$ is uniserial and hence the non-zero submodule $Rx = R(1-a)x$ is essential in $R(1-a)m$. Because $PK = 0$, the module K is semisimple and hence $K = Rx \oplus L$, for some submodule L of K. But this implies that K is essential in the submodule $R(1-a)m \oplus L$ of M so that $(1-a)m \in K$. Now $x = (1-a)x = (1-a)bm \in PK = 0$, a contradiction. Thus $K \cap PM = 0$, as required. □

Lemma 4.5. *Let R be a ring such that every essential ideal contains a product of almost principal maximal ideals and let K be a submodule of an R-module M such that $PK = K \cap PM$ for every almost principal maximal ideal P of R. Then K is a closed submodule of M.*

Proof. By Zorn's Lemma there exists a submodule N of M containing K such that K is an essential submodule of N and N is a closed submodule of M. Let $x \in N$. We shall show that $x \in K$. There exists an essential ideal Q of R such that $Qx \subseteq K$. By hypothesis Q contains a product of almost principal maximal ideals and hence we can suppose without loss of generality that Q is an almost principal maximal ideal of R. By Lemma 4.4, $QN = N \cap QM$ and by hypothesis $QK = K \cap QM$ so that $QK = K \cap QM = K \cap N \cap QM = K \cap QN$. There exist elements a, b in Q such that $(1-a)Q \subseteq Rb$. Now $bx \in K \cap QN = QK$ so that $(1-a)bx \in bK$. There exists $u \in K$ such that $(1-a)bx = bu$ and hence $b[(1-a)x - u] = 0$. This implies that $Q(1-a)[(1-a)x - u] = 0$ and hence the element $(1-a)[(1-a)x - u]$ belongs to the socle of N. But K is essential in N, so that $(1-a)[(1-a)x - u] \in K$. It follows that $(1-a)^2x \in K$. But $ax \in K$ so that $x \in K$. We have proved that $K = N$ and hence K is a closed submodule of M. $\square$

Combining Lemmas 4.4 and 4.5 it gives Proposition 4.1. Note that Proposition 4.1 has the following immediate consequence.

Corollary 4.6. *Let R be a Dedekind domain. Then a submodule K of an R-module M is closed in M if and only if K is a $\mathcal{P}$-pure submodule of M.*

Note that if S is a Dedekind domain and X is any non-zero injective S-module then the trivial extension R of X by S is a ring such that every maximal ideal is locally principal. However, if A is the ideal of R consisting of all elements $(0, x)$ for x in X then A is an essential ideal of R such that $PA = A \cap PR$ for every maximal ideal P of R. Note further that the ideal A does not contain a product of maximal ideals of R.

Let R be any ring. An R-module X is called *c-injective* provided, for every R-module M, every homomorphism $\varphi : K \to X$ from any closed submodule K of M to X can be lifted to M. In [17] it is proved that if R is a Dedekind domain then every direct product of simple R-modules is c-injective. We aim next to characterize all c-injective modules over Dedekind domains. First note the following result.

Lemma 4.7. *Let R be a Dedekind domain. Then the following statements are equivalent for an R-module X.*

(i) *X is c-injective.*
(ii) *X is $\mathcal{P}$-pure-injective.*
(iii) *X is isomorphic to a direct summand of a direct product of simple R-modules and injective R-modules.*

Proof. (i) $\Leftrightarrow$ (ii). By Corollary 4.6.
(ii) $\Leftrightarrow$ (iii). By Theorem 2.8. $\square$

Let R be a ring. A submodule K of a (left) R-module M is called *pure* provided for every (finitely presented) right R-module U, the induced homomorphism $U \otimes_R K \to U \otimes_R M$ of Abelian groups is a monomorphism. When R is a Dedekind domain (more generally a Prüfer domain), a submodule K of an R-module M is *pure* if and only if $K \cap aM = aK$ for all $a \in R$. An R-module X is called *pure-injective* if for every R-module M and every pure submodule K of M, every homomorphism $\varphi : K \to X$ lifts to M. Note the following result.

Lemma 4.8. *Let R be a Dedekind domain and let X be any R-module such that $aX = 0$ for some $0 \neq a \in R$. Then X is pure-injective.*

Proof. By Lemma 2.1 and [10, Theorem 5]. $\square$

Let R be a Dedekind domain and let X be a c-injective R-module. By Lemma 4.7 there exists an R-module Y such that Y is a direct product of simple modules and injective modules and X is isomorphic to a direct summand of Y. Note that if M_j $(j \in J)$ are R-modules such that $PM_j = 0$ $(j \in J)$ then $P \prod_{j \in J} M_j = 0$. It follows that $Y = Y_0 \oplus (\prod_{i \in I} Y_i)$ is a direct product of an injective submodule Y_0 and homogeneous semisimple R-modules Y_i $(i \in I)$ such that $P_i Y_i = 0$ for some maximal ideal P_i of R for each $i \in I$ and $P_i \neq P_j$ for all $i \neq j$ in I. By [10, Theorem 8], $X = X_0 \oplus X'$ for some injective submodule X_0 and some submodule X' which is isomorphic to a direct summand of $\prod_{i \in I} Y_i$.

Let $Y' = \prod_{i \in I} Y_i$. Without loss of generality we can suppose that $X' \subseteq Y'$. Note that Y' is a reduced R-module (i.e. Y' does not contain any non-zero injective submodule). For each $i \in I$, Lemma 4.8 gives that Y_i is pure-injective and hence Y' and X' are also pure-injective. By [6, Proposition XIII.4.5] (or see [19]) the reduced pure-injective module X' can be written as $X' = \prod_{P \in \Pi} A_P$, for some collection Π of maximal ideals of R where A_P is a module over the localization R_P of R at the maximal ideal P for each $P \in \Pi$.

Let $P \in \Pi$. Suppose that $P \neq P_i$ for all $i \in I$. Let $i \in I$ and let $c \in P_i \backslash P$. Then $A_P = cA_P$ so that $A_P \subseteq \prod_{j \neq i} Y_j$. It follows that $A_P \subseteq \bigcap_{i \in I} [\prod_{j \neq i} Y_j]$ and therefore in this case $A_P = 0$. Thus if $A_P \neq 0$ then $P = P_i$ for some $i \in I$, $A_P \subseteq Y_i$ and hence $PA_P = 0$. We have proved that $PA_P = 0$ for all $P \in \Pi$. Thus we have the following generalization of [8, Lemma 4].

Theorem 4.9. *Let R be a Dedekind domain. Then an R-module X is c-injective if and only if X is a direct product of homogeneous semisimple modules and injective modules.*

5. More on Dedekind domains

We shall show in this section that Theorem 4.9 does not extend to commutative Noetherian domains which are not Dedekind. In fact we prove that if R is a commutative Noetherian domain and P a maximal ideal of R then the simple R-module R/P is c-injective if and only if the ideal P is invertible. In particular, note that invertible prime ideals of R have height 1 so that for many commutative Noetherian rings R no simple R-module is c-injective.

Lemma 5.1. *Let R be a commutative Noetherian local ring with unique maximal ideal P and let M be the free R-module $R \oplus R$. Then a submodule K of M is a direct summand of M if and only if $PK = K \cap PM$.*

Proof. The necessity is clear. Conversely, suppose that $PK = K \cap PM$. Suppose that $K \subseteq PM$. Then $K = PK$ so that $K = 0$ by Nakayama's Lemma. Suppose that $K \nsubseteq PM = P \oplus P$. Without loss of generality, $(1, a) \in K$ for some $a \in R$. Then $M = R(1, a) \oplus (0 \oplus R)$ so that $K = R(1, a) \oplus L$ where L is the submodule $K \cap (0 \oplus R)$ of the direct summand $0 \oplus R$. Since L is a direct summand of K it follows that $PL = L \cap PM$. Clearly it is sufficient to prove that L is a direct summand of $H = 0 \oplus R$. But $PL = L \cap PH$ and hence $L = 0$ or $L = H$. Thus L is a direct summand of H, as required. □

Corollary 5.2. *Let R be a commutative Noetherian local domain with unique maximal ideal P and let M be the R-module $R \oplus R$. Suppose that $PK = K \cap PM$ for every closed submodule K of M. Then R is a discrete valuation ring.*

Proof. By Lemma 5.1 the R-module M is an extending module. Then [3, Corollary 12.10] gives that R is a Prüfer domain and hence a Dedekind domain. Because R is local, the ring R is a DVR. □

Now let R be a commutative Noetherian domain and let M be the free R-module $R \oplus R$. Let P be a maximal ideal of R. Suppose further that

$$PK = K \cap PM$$

for every closed submodule K of M. Form the commutative Noetherian local domain R_P and the R_P-module $M' = M_P$. Let K' be any closed submodule of M'. Note that M is isomorphic to the R-submodule $M^* = \{m/1\colon\ m \in M\}$ of M_P. Because M' is a torsion-free R_P-module, the factor module M'/K' is torsion-free. If $K^* = \{x/1\colon\ x \in K'\}$ then M^*/K^* is a torsion-free R-module so that K^* is a closed submodule of M^*. By our above assumption, $PK^* = K^* \cap PM^*$ and it easily follows that $R_P PK' = K' \cap R_P PM'$. By Corollary 5.2 the ring R_P is a DVR. In particular, there exists an element $a \in P$ such that $R_P P = R_P a$. It is now easy to show that there exists $b \in P$ such that $(1 - b)P \subseteq Ra$. Thus P is almost principal. But R is a domain and hence P is an invertible ideal of R by [4, Theorem 1.2 and Lemma 3.6]. We have proved the following result.

Lemma 5.3. *Let R be a commutative Noetherian domain, let P be a maximal ideal of R and let M be the free R-module $R \oplus R$. Suppose that $PK = K \cap PM$ for every closed submodule K of M. Then P is an invertible ideal of R.*

This brings us to the main result of this section.

Theorem 5.4. *Let R be a commutative Noetherian domain and let P be a maximal ideal of R. Then the following statements are equivalent.*

(i) *The module R/P is c-injective.*
(ii) *The module R/P is c-M-injective, where M is the free R-module $R \oplus R$.*
(iii) *P is an invertible ideal.*

Proof. (i) $\Rightarrow$ (ii). Clear.
(ii) $\Rightarrow$ (iii). By Lemmas 2.4 and 5.3.
(iii) $\Rightarrow$ (i). By Lemmas 4.4 and 2.4. □

Corollary 5.5. *A commutative Noetherian domain R is Dedekind if and only if every simple R-module is c-injective.*

Proof. By Theorem 5.4 and [20, Theorem 12, p. 275]. □

Let R denote the polynomial ring in indeterminates x and y over a field. Then R is a commutative Noetherian domain and no maximal ideal of R is invertible. Does this mean that every c-injective module is injective? We do not know the answer to this question.

Acknowledgments

The authors are extremely grateful to Professor W.J. Wickless for showing them how to prove Theorem 4.9, although their proof is different.

Some of this work appears in the PhD thesis of the first author who would like to thank his supervisor Professor Rafail Alizade. The first author has also been supported by TÜBİTAK by the project numbered 107T709. Some of this work was carried out during a visit of the second author to the University of Glasgow in February 2007 and a visit of the third author to the İzmir Institute of Technology in November 2006 and to the University of Lisbon in May 2007. The second author would like to thank the University of Glasgow for their hospitality. Her work was carried out within the project POCTI-ISFL-1-143 of CAUL, financed by FCT and FEDER. The third author would like to thank the İzmir Institute of Technology for their hospitality and TÜBİTAK for financial support, also the Center of Algebra of the University of Lisbon for their hospitality and for their financial support.

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